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SPECIAL ISSUE

Optimal Robust Tracking of a Discrete Minimum-Phase Plant under the Unknown Bias and Norm of an External Disturbance and the Unknown Norm of Uncertainties

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Abstract—This paper addresses a problem of the optimal robust tracking of a given bounded reference signal for a discrete-time minimum-phase plant with a known approximate nominal model under a bounded and biased external disturbance and coprime factor perturbations. The bias and norm of the external disturbance and the gains of the perturbations are assumed to be unknown. The control criterion is the worst-case asymptotic tracking error in the class of the disturbances and perturbations under consideration, which depends on the above unknown parameters and the reference signal. A solution of the optimal tracking problem with a given accuracy is based on optimal errors quantification within the ℓ_1-theory of robust control, polyhedral estimation of the unknown parameters, and treating the control criterion as the identification criterion.

Keywords: robust control, optimal control, bounded disturbance, uncertainty, errors quantification, set-membership approach

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1. INTRODUCTION

This paper addresses the optimal tracking problem of a linear discrete dynamic plant with a given and tested transfer function. By assumption, the plant is affected by a bounded external disturbance with an unknown bias and unknown bounds and by perturbations (uncertainties) for its output and control with unknown norms (gains). The problem is addressed within the ℓ_1-theory of robust control, laid down in [1, 2] and corresponding to the signal space ℓ_∞ of bounded real sequences. The problem has the following difficulty: to minimize a criterion in the form of the worst-case asymptotic tracking error in the class of admissible disturbances, it is necessary to compensate for the unknown bias and justify an optimal estimator for the criterion under the non-identifiability of all the unknown parameters mentioned above.

The solution of the optimal tracking problem described is based on optimal errors quantification using the set-membership approach and treating the control criterion as an ideal identification criterion. The set-membership approach in system identification, initially involving the assumption of known upper bounds on deterministic disturbances, gained wide popularity in the late 1980s and was reduced to the development of computable upper and lower approximations (ellipsoids, parallelotopes, etc.) of parameter sets consistent with measurement data. (Here, we refer to the first special issues of two leading journals on control theory [3, 4].) Applications of these approximations to control problems are rarely described and are accompanied by various additional assumptions, such as a priori known stabilizing control. This approach is criticized by supporters of stochastic disturbance models for its conservatism caused by a priori assumptions on known upper bounds on disturbances. In parallel, active research in the field of identification for robust control and uncertainty quantification began in the early 1990s. A decade and a half later, it was noted in the review [5] that estimation of uncertainty sets was often mistakenly attributed to identification for control, as in most of the corresponding studies, the control objective was not considered during identification. The problems of model verification and uncertainty estimation remain topical to the present time [6, 7], but are still considered mainly beyond the context of control problems and with artificial criteria motivated by the objectives of identification itself.

In this paper, bias estimation and errors quantification are based on the set-membership approach and treating the control criterion as an ideal identification criterion. The potential applicability of such a combined framework arises from two circumstances. First, in the ℓ_1-theory of robust control, explicit representations are obtained for asymptotic performance indices in terms of induced norms of the transfer functions of a closed-loop control system and the norms of all disturbances and uncertainties [8–11]. Second, the bounded disturbance model allows for the direct use of current measurement data for online model verification [12]. In the general case, such an approach to control-oriented identification is computationally intractable due to the complexity of computing current optimal estimates. But it is computationally tractable in the case of linear or linear-fractional, with respect to the estimated parameters, performance indices [13]. In the problem under consideration, the performance index (control criterion) is a non-convex quadratic-fractional function of the unknown parameters (see the representation (9)). For a known bias, the control criterion becomes linear-fractional, and the problem of errors quantification for this case was solved in [14], where the idea of estimating the unknown bias using a grid of test values was also formulated. Below, we rigorously justify this idea and prove a rigorous result on the solution of the asymptotically optimal tracking problem with a given accuracy. Simulation results and related remarks illustrate the effectiveness of the solution proposed.

Notation:
$|\varphi|$ is the Euclidean norm of a vector $\varphi \in \mathbb{R}^n$;
$x_s^t = (x_s, x_{s+1}, \ldots, x_t)$ for a real sequence $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$;
$|x_s^t| = \max_{s \leqslant k \leqslant t} |x_k|$;
$\|x\|_{ss} = \limsup_{t \to +\infty} |x_t|$;
$\|x\|_\infty = \sup_t |x_t|$ is the norm in the space ℓ_∞ of bounded sequences;
$\|x\|_1 = \sum_{k=0}^{+\infty} |x_k|$ is the norm in the space ℓ_1 of absolutely summable sequences;
$\|G\| = \sum_{k=0}^{+\infty} |g_k| = \|g\|_1$ is the induced norm of a stable linear time-invariant causal system $G : \ell_\infty \to \ell_\infty$ with a transfer function $G(\lambda) = \sum_{k=0}^{+\infty} g_k \lambda^k$.

2. THE PLANT MODEL AND MEANINGFUL PROBLEM STATEMENT

The plant model is described by the equation

$$a(q^{-1}) y_t = b(q^{-1}) u_t + v_t, \quad t = 1, 2, 3, \ldots, \tag{1}$$

where $y_t \in \mathbb{R}$ is the measured output of the plant at a time instant t, $u_t \in \mathbb{R}$ is the control input, $v_t \in \mathbb{R}$ is a total disturbance, and q^{-1} is the backward shift operator ($q^{-1} y_t = y_{t-1}$). The initial conditions $y_{1-n}^0 = (y_{1-n}, \ldots, y_0)$ are arbitrary, and $u_t = 0$ for $t \leqslant 0$. The polynomials

$$a(\lambda) = 1 + a_1 \lambda + \ldots + a_n \lambda^n, \quad b(\lambda) = b_1 \lambda + \ldots + b_m \lambda^m$$

characterize the **nominal model** of the plant, i.e., the model without the disturbance v. The total disturbance v has the form

$$v_t = c^w + \delta^w w_t + \delta^y \Delta^1(y)_t + \delta^u \Delta^2(u)_t, \quad \|w\|_\infty \leqslant 1,\ \delta^w > 0,\ \delta^y > 0,\ \delta^u > 0. \tag{2}$$

The parameters c^w and δ^w in (2) characterize the bias of the external disturbance $c^w + \delta^w w_t$ and the upper bound on the unbiased disturbance $\delta^w w$, respectively. The numbers $\delta^y > 0$ and $\delta^u > 0$ are the gains (induced norms) of the perturbations affecting the output and control, respectively, and

$$|\Delta^1(y)_t| \leqslant p_t^y = \max_{t-\mu\leqslant k\leqslant t-1} |y_k|, \quad |\Delta^2(u)_t| \leqslant p_t^u = \max_{t-\mu\leqslant k\leqslant t-1} |u_k|. \tag{3}$$

In the ℓ_1-theory of robust control, these perturbations are called uncertainties with limited memory μ, which ensures the independence of the asymptotic dynamics of the closed-loop control system from the initial data. The uncertainty memory μ is chosen by the designer to be arbitrarily large, but not infinite, without compromising the guaranteed control performance. The description of disturbances in the form (2), (3) is equivalent [10, 11] to the inequalities

$$|v_t - c^w| \leqslant \delta^w + \delta^y p_t^y + \delta^u p_t^u \quad \forall t. \tag{4}$$

A priori information about the plant is contained in the following assumptions.

A1. The polynomials $a(\lambda)$ and $b(\lambda)$ of the nominal plant are known, $b_1 \neq 0$.

A2. The roots of the polynomial $\frac{b(\lambda)}{\lambda}$ lie outside the closed unit circle of the complex plane.

A3. The parameter column vector $\delta = (\delta^w, \delta^y, \delta^u)^{\mathrm{T}}$ is unknown, and the bias $c^w \in [c^w_{\min}, c^w_{\max}]$ is unknown, albeit with given $c^w_{\min}$ and $c^w_{\max}$.

A4. The asymptotic upper bound $\|r\|_{ss}$ of the reference signal r or its upper bound is known.

Assumption A1 also covers the case when the "true" nominal model is unknown and its estimator, obtained by some identification method, is available for testing. Assumption A2 ensures the boundedness of the control input u if the plant output y is bounded. (Such a plant is called minimum-phase.) Assumption A4 will be commented upon after the rigorous formulation of the problem at the end of Section 3. Another mandatory assumption restricting the norms of the perturbations will be introduced in Section 3 after Theorem 1.

Meaningful problem statement: it is required to design a control law minimizing the worst asymptotic tracking error of a given bounded signal for a set of disturbances satisfying inequalities (4).

To solve the optimal problem with a given accuracy, one needs to quantify the errors online (i.e., determine their unknown parameters δ) to estimate the tracking performance and compensate for the unknown bias c^w.

3. THE TRACKING PERFORMANCE OF AN OPTIMAL CONTROLLER UNDER A KNOWN BIAS c^w. PROBLEM STATEMENT

Let $r = (r_1, r_2, r_3, \ldots)$ be a given bounded signal ($r \in \ell_\infty$). The control criterion of the tracking problem has the form

$$J_\mu(c^w, \delta) = \sup_{v\in V} \|y - r\|_{ss}, \quad \|y - r\|_{ss} := \limsup_{t\to+\infty} |y_t - r_t|, \tag{5}$$

where V is the set of all disturbances v satisfying inequalities (4).

Consider a controller described by the equation

$$b(q^{-1})u_t = (a(q^{-1}) - 1)y_t + r_t - c^w. \tag{6}$$

Note that (6) specifies the value of u_{t-1}, not u_t, which does not figure in this equation. For the output of the closed-loop control system (1), (6), we then obtain

$$y_t - r_t = v_t - c^w = \delta^w w_t + \delta^y \Delta^1(y)_t + \delta^u \Delta^2(u)_t. \tag{7}$$

Due to the arbitrariness and unpredictability of the right-hand side in (7), the controller (6) is **optimal** for the control criterion (5).

Definition 1. The closed-loop system (1), (6) is said to be robustly stable in the class of disturbances V if $J_\mu(c^w, \delta) < +\infty$.

To formulate a theorem on the performance of the optimal controller (6), we denote its transfer functions relating y and r to the control input u :

$$G_{uy}(\lambda) = \frac{a(\lambda) - 1}{b(\lambda)}, \quad G_{ur}(\lambda) = \frac{1}{b(\lambda)}.$$

Theorem 1. *Under Assumptions* A1 *and* A2*, the following assertions are true.*

1) *The closed-loop system* (1), (6) *is robustly stable in the class* V *with a disturbance memory* $\mu = +\infty$ *if and only if*

$$\delta^y + \delta^u \|G_{uy}\| < 1. \tag{8}$$

For the system with the zero initial conditions y_{1-n}^0 *and* $\mu = +\infty$,

$$J(c^w, \delta) := J_{+\infty}(c^w, \delta) = \frac{\delta^w + \delta^y \|r\|_{ss} + \delta^u(|c^w| + \|r\|_{ss})\|1/b(q^{-1})\|}{1 - \delta^y - \delta^u \|G_{uy}\|}. \tag{9}$$

2) *For the system with arbitrary initial conditions* y_{1-n}^0 *and* $\mu < +\infty$,

$$J_\mu(c^w, \delta) \leqslant J(c^w, \delta) \quad \forall\, \mu > 0, \tag{10}$$

and if the sequence $|r|$ *uniformly often falls into the neighborhood of the upper limit* $\|r\|_{ss}$ *(see the definition in* [10]*), then for any initial conditions*

$$J_\mu(c^w, \delta) \nearrow J(c^w, \delta) \quad (\mu \to +\infty), \tag{11}$$

where the sign $\nearrow$ *means monotonic convergence from below as* $\mu \to +\infty$.

The proof of Theorem 1 was given in [14].

The final assumption (A5) restricting the norms of the perturbations follows from Theorem 1.

A5. A number $\bar{\delta}$ is known such that

$$\delta^y + \delta^u \|G_{uy}\| \leqslant \bar{\delta} < 1. \tag{12}$$

Assumption A5 is not restrictive compared to the robust stability condition (8). According to the meaning of the problem, the parameter $\bar{\delta}$ is assigned by the designer and can be chosen arbitrarily close to 1. But for values of $\delta^y + \delta^u \|G_{uy}\|$ close to 1, the control criterion $J_\mu(c^w, \delta)$ becomes too large and the nominal model with the given tested polynomials $a(\lambda)$ and $b(\lambda)$ or the plant with such perturbations can be considered unacceptable.

Problem statement. Under a priori information A1–A5 and a given tracking signal r, it is required to design a feedback control law $u_t = U_t(y_1^t, u_1^{t-1})$ (with finite memory) that ensures the inequality

$$\|y - r\|_{ss} \leqslant J(c^w, \delta) \tag{13}$$

with a given accuracy.

The main difficulty of the problem is to ensure inequality (13) under the non-identifiability of c^w and δ (see subsection 4.1).

The index (9), used below as an identification criterion, depends on $\|r\|_{ss}$. If this value is a priori unknown, the recursively computable non-decreasing estimators $R_t = \max_{1 \leqslant k \leqslant t} |r_k| \leqslant \|r\|_\infty$ can be used instead to obtain a fundamentally theoretically unimprovable tracking performance guarantee with $\|r\|_{ss}$ replaced by $\|r\|_\infty$.

4. OPTIMAL TRACKING

The solution of the problem is based on optimal errors quantification for the nominal model being tested.

4.1. Optimal Errors Quantification with a Known Bias c^w

Due to the plant equation (1) and inequalities (4), given a known bias c^w, complete information about the unknown δ at a time instant t is contained in the a priori assumption A5 and the inclusion

$$\delta \in D_t = \left\{\hat{\delta} \geqslant 0 \mid |a(q^{-1})y_k - b(q^{-1})u_k - c^w| \leqslant \hat{\delta}^w + \hat{\delta}^y p_k^y + \hat{\delta}^u p_k^u \quad \forall k \leqslant t\right\}, \tag{14}$$

where $\hat{\delta} = (\hat{\delta}^w, \hat{\delta}^y, \hat{\delta}^u)^{\mathrm{T}}$. The system of inequalities in (14) is equivalent to the description of system (1)–(4) on the interval $[1, t]$ for any control u_0^{t-1}. Then the best estimator of the parameter δ in terms of the control criterion J, consistent with the measurements y_0^t and u_0^{t-1}, has the form

$$\delta_t = \underset{\hat{\delta} \in D_t}{\operatorname{argmin}}\, J(c^w, \hat{\delta}). \tag{15}$$

The optimal problem (15) is a linear-fractional programming problem with the unknown row vector $\hat{\delta}$. It is reduced to a linear programming problem in the standard way by introducing an additional real variable [15]. The number of linear inequalities with respect to $\hat{\delta}$ in the description of the sets D_t can infinitely increase as t grows. To ensure the boundedness of the number of inequalities and the convergence of the polyhedral and vector estimators of the unknown column vector δ in finite time, we choose the parameter $\varepsilon_1 > 0$, which specifies the dead zone size when updating the estimators. The initial polyhedral estimator of δ has the form

$$P_0 = \left\{\hat{\delta} = (\hat{\delta}^w, \hat{\delta}^y, \hat{\delta}^u)^{\mathrm{T}} \mid \hat{\delta} \geqslant 0,\ \hat{\delta}^y + \hat{\delta}^u \|G_{uy}\| \leqslant \bar{\delta} < 1\right\}, \quad \delta_0 = (0, 0, 0)^{\mathrm{T}}.$$

Denoting

$$\nu_{t+1} = |a(q^{-1})y_{t+1} - b(q^{-1})u_{t+1} - c^w|, \quad \phi_{t+1} = (1, p_{t+1}^y, p_{t+1}^u)^{\mathrm{T}}, \tag{16}$$

we write the new inequality in the description of D_{t+1} as

$$\delta \in \Omega_{t+1} = \left\{\hat{\delta} \mid \nu_{t+1} \leqslant \hat{\delta}^{\mathrm{T}} \phi_{t+1}\right\}. \tag{17}$$

Let P_t and δ_t be the polyhedral and vector estimators of δ at a time instant t. We set

$$P_{t+1} = \begin{cases} P_t & \text{if } \nu_{t+1} \leqslant \delta_t^{\mathrm{T}} \phi_{t+1} + \varepsilon_1 |\phi_{t+1}| \\ P_t \cap \Omega_{t+1} & \text{otherwise,} \end{cases} \tag{18}$$

$$\delta_{t+1} = \underset{\hat{\delta} \in P_{t+1}}{\operatorname{argmin}}\, J(c^w, \hat{\delta}). \tag{19}$$

According to (18), the polyhedral estimator P_{t+1} is updated by adding a new inequality only if the distance from δ_t to the half-space $\Omega_{t+1} \subset \mathbb{R}^3$ exceeds ε_1. Note that all estimators P_t are unbounded in the direction of growth of the variable $\hat{\delta}^w$.

4.2. Optimal Tracking under an Unknown Bias c^w

To compensate for the unknown bias $c^w \in [c_{\min}^w, c_{\max}^w]$, we will estimate it using a grid of the form

$$c_k^w = c_{\min}^w + k\varepsilon_2, \quad k = 0, 1, \ldots, N, \qquad \varepsilon_2 = \frac{c_{\max}^w - c_{\min}^w}{N} > 0, \tag{20}$$

which yields a guaranteed estimator of the bias c^w with the desired accuracy $\varepsilon_2/2$ by choosing a sufficiently large N. For each bias c^w_k and each time instant t, we compute the polyhedral $P_{k,t}$ and vector $\delta_{k,t}$ estimators of the unknown vector δ. We define the best estimate number k_t of the vector δ at a time instant t by the formula

$$k_t = \operatorname*{argmin}_k J(c^w_k, \delta_{k,t}). \tag{21}$$

The control input u_t at a time instant t is determined by the *adaptive controller* corresponding to this estimate:

$$b(q^{-1})u_{t+1} = (a(q^{-1}) - 1)y_{t+1} + r_{t+1} - c^w_{k_t}. \tag{22}$$

After measuring the output y_{t+1}, the residuals

$$\nu_{k,t+1} = |a(q^{-1})y_{t+1} - b(q^{-1})u_{t+1} - c^w_k|$$

and the estimators $P_{k,t+1}$ and $\delta_{k,t+1}$ for $k = 0, 1, \ldots, N$ are computed according to (16)–(19). (The corresponding formulas, with the subscript k in each, are omitted here for brevity.)

Theorem 2. *Under Assumptions* A1–A5, *let the plant* (1) *be regulated by the adaptive controller* (22) *with the estimator* (16)–(19), (21) *and the dead zone parameter* ε_1 *in* (18) *such that*

$$0 < \varepsilon_1 < \frac{1 - \bar{\delta}}{1 + \|G_{uy}\|}. \tag{23}$$

Then the number of updates in the polyhedral $P_{k,t}$ *and vector* $\delta_{k,t}$ *estimators is finite for all* $k \in \{0, 1, \ldots, N\}$, *and the tracking error satisfies the inequality*

$$\|y - r\|_{ss} \leqslant J(c^w_{k_\infty}, \delta_\infty + \varepsilon_1(1,1,1)^{\mathrm{T}}) = J(c^w_{k_\infty}, \delta_\infty) + O(\varepsilon_1) \quad (as\ \varepsilon_1 \to 0), \tag{24}$$

where k_∞ *is the final value of the best estimate number* (21) *for the unknown* δ, δ_∞ *is the final value of* $\delta_{k_\infty,t}$, *and*

$$J(c^w_{k_\infty}, \delta_\infty) \leqslant J\left(c^w, \delta + \left(\frac{\varepsilon_2}{2} + \varepsilon_1, \varepsilon_1, \varepsilon_1\right)\right) = J(c^w, \delta) + O(\varepsilon_1 + \varepsilon_2) \quad (\varepsilon_1 + \varepsilon_2 \to 0). \tag{25}$$

Proof. For any control u_t and any $k \in \{0, 1, \ldots, N\}$, from the plant equation (1) and the representation (4) of the total disturbance v it follows that

$$|a(q^{-1})y_{t+1} - b(q^{-1})u_{t+1} - c^w_k| \leqslant |c^w - c^w_k| + \delta^w + \delta^y p^y_{t+1} + \delta^u p^u_{t+1} \quad \forall\, t. \tag{26}$$

By the representation (4), inequalities (26) allow treating the plant as a virtual object of the form (1), in which the virtual external disturbance has the bias c^w_k and the norm of the unbiased external disturbance does not exceed

$$\bar{\delta}^w_k = |c^w - c^w_k| + \delta^w. \tag{27}$$

We prove that the number of updates in the estimators $P_{k,t}$ and $\delta_{k,t}$ is finite for all k. For each update of the estimators, according to (18), we have

$$\varepsilon_1|\phi_{t+1}| < \nu_{t+1} - \delta_t^{\mathrm{T}}\phi_{t+1}.$$

Then, for any $\hat{\delta} \in \Omega_{t+1}$, (17) implies

$$\varepsilon_1|\phi_{t+1}| < |(\hat{\delta} - \delta_t)^{\mathrm{T}}\phi_{t+1}| \leqslant |\hat{\delta} - \delta_t||\phi_{t+1}|$$

and, consequently, $|\hat{\delta} - \delta_t| > \varepsilon_1$. Hence, for all k, the distance from the estimator $\delta_{k,t}$ to the half-space $\Omega_{k,t+1}$ is greater than ε_1. As $P_{k,t+1} \subset \Omega_{t+1}$, the distance from $\delta_{k,t}$ to $P_{k,t+1}$ is also greater than ε_1. The polyhedral estimators $P_{k,t}$ decrease monotonically in time due to the addition of new inequalities. Moreover, the balls of radius $\varepsilon_1/2$ centered at $\delta_{k,t}$ have empty intersection with similar balls centered at the future updated estimators $\delta_{k,s}$ (for $s > t$) and, consequently, with all balls $\delta_{k,s}$ for $s \neq t$. It follows that the number of possible updates in the estimators $\delta_{k,t}$ is finite for all k, since they all lie in the corresponding bounded sets $\{\hat{\delta}_k \mid J(c_k^w, \hat{\delta}_k) \leqslant J(c_k^w, (\bar{\delta}_k^w, \delta^y, \delta^u)^{\mathrm{T}})\}$, where $\bar{\delta}_k^w$ is given by (27).

We denote by $\delta_{k,\infty} = (\delta_{k,\infty}^w, \delta_{k,\infty}^y, \delta_{k,\infty}^u)^{\mathrm{T}}$ the final values, i.e., the limit values of the estimators $\delta_{k,t}$ achieved in a finite time $t_{k,\infty}$, and set $t_\infty = \max_k t_{k,\infty}$. Then $\delta_{k,t} = \delta_{k,\infty}$ for all $t \geqslant t_\infty$ and all k.

Let k_∞ be the steady-state number of the best estimate of the vector δ :

$$k_\infty = \underset{k}{\operatorname{argmin}}\, J(c_k^w, \delta_{k,\infty}).$$

Due to (21), we have

$$J(c_{k_\infty}^w, \delta_{k_\infty}) \leqslant J(c_k^w, \delta_{k,\infty}) \quad \forall k. \tag{28}$$

For all $t \geqslant t_\infty$, in view of (18), the residuals (26) in the closed-loop adaptive system with the steady-state controller satisfy

$$\nu_{k_\infty,t} \leqslant \delta_{k,\infty}^{\mathrm{T}} \phi_t + \varepsilon_1 |\phi_t| \leqslant (\delta_{k_\infty}^{\mathrm{T}} + \varepsilon_1(1,1,1))\phi_t. \tag{29}$$

By Theorem 1, this inequality implies (24).

We denote by

$$k_* = \underset{k}{\operatorname{argmin}}\, |c^w - c_{k_\infty}^w|$$

the number of the estimate c_k^w closest to c^w. Then $|c^w - c_{k_*}^w| \leqslant \varepsilon_2/2$ and, due to (18),

$$\begin{aligned} |a(q^{-1})y_{t+1} - b(q^{-1})u_{t+1} - c_{k_*,t}^w| &\leqslant \frac{\varepsilon_2}{2} + \delta^{\mathrm{T}} \phi_{t+1} + \varepsilon_1 |\phi_t| \\ &\leqslant \delta + \left(\frac{\varepsilon_2}{2} + \varepsilon_1, \varepsilon_1, \varepsilon_1\right)\Big)^{\mathrm{T}} \phi_{t+1} \end{aligned} \tag{30}$$

for all $t \geqslant t_\infty$. According to Theorem 1, this inequality yields

$$J(c_{k_*}^w, \delta_{k_*,\infty}) \leqslant J\left(c^w, \delta + \left(\frac{\varepsilon_2}{2} + \varepsilon_1, \varepsilon_1, \varepsilon_1\right)\right). \tag{31}$$

Using (28) with $k = k_*$ and (31), we obtain inequality (25). Finally, the term $O(\varepsilon_1 + \varepsilon_2)$ in (25) follows from the fact that $J(c^w, \delta)$ is a fractional rational function of δ and its denominator is separated from 0 by Assumption A5. The proof of Theorem 2 is complete.

Remark 1. Inequalities (24) and (25) mean the suboptimality of the solution of the tracking problem (13). The estimate $O(\varepsilon_1 + \varepsilon_2)$ of the solution accuracy in the stated optimal problem, guaranteed by inequality (25), is only qualitative and cannot be used for computations since c^w and δ are unknown. For a particular realization of the disturbances, the best computable estimate of the solution accuracy in the optimal tracking problem is the current difference

$$(\Delta J)_t = J(c_{k_t}^w, \delta_{k_t,t} + \varepsilon_1(1,1,1)^{\mathrm{T}}) - J(c_{k_t}^w, \delta_{k_t,t}), \tag{32}$$

which is consistent with the measured data y_{1-n}^t, u_1^t and will be guaranteed as the estimators converge in a finite time. Although the convergence time of the estimators to the final value is unknown, a long period of unchanged estimates actually confirms the validity of this estimate by Theorem 2. If the current estimate of the solution accuracy is unsatisfactory, one can (at any time) decrease the dead zone parameter ε_1 to improve the accuracy. In this case, the number of updates in the estimators $P_{k,t}$ and $\delta_{k,t}$ may increase. The grid step ε_2 has a more transparent impact on the optimization accuracy (the term $\varepsilon_2/2$ is added to the estimates $\delta^w_{k,t}$) and can be chosen a priori, while keeping in mind that a decrease in the grid step ε_2 will cause an increase in the number of polyhedral $P_{k,t}$ and vector $\delta_{k,t}$ estimators computed in parallel.

5. SIMULATION

Let the plant be described by equation (1) with the unknown parameters

$$\theta^* = [a_1^*, a_2^*, b_1^*, b_2^*, b_3^*] = [-2.7;\ 1.8;\ 2;\ -3.36;\ 1.4] \tag{33}$$

of the nominal model, and let a nominal model with poles 0.7 and 0.8 (the roots of $a(\lambda)$) and zeros 1.1 and 1.3 (the roots of $b(\lambda)$) and the coefficient $b_1 = 2$ be available for testing. This nominal model matches an unstable minimum-phase plant (1) with the parameters

$$\theta = [a_1, a_2, b_1, b_2, b_3] = [-2.6786;\ 1.7857;\ 2;\ 3.3566;\ 1.3986], \tag{34}$$

slightly differing from the parameters (33). Let the plant with the parameters (33) be regulated by the adaptive controller (22), which is optimal for the tested plant with the parameters (34). The characteristic polynomial of this closed-loop system has roots $0.7512 \pm 8.9242i, 1.3032$, and 1.0945 (with an accuracy of 10^{-4}), being greater than 1 by absolute value; therefore, the closed-loop system without the perturbations is stable. The dynamics of this closed-loop system can be treated as the dynamics of that with a plant with nominal parameters θ and additional relatively small perturbations

$$\Delta(y_{t-2}^{t-1}, u_{t-2}^{t-1}) = (a_1 - a_1^*)y_{t-1} + (a_2 - a_2^*)y_{t-2} + (b_1^* - b_1)u_{t-1} + (b_1^* - b_1)u_{t-2},$$

arising from the "inaccurate" coefficients of the nominal model tested. The disturbance v_t in the nominal model with the parameters θ is described by

$$v_t = c^w + \delta^w w_t + k_t^y\, \delta^y\, |y_{t-\mu}^{t-1}| + k_t^u\, \delta^u |\, u_{t-\mu}^{t-1}|,\ |k_t^y| \leqslant 1,\ |k_t^u| \leqslant 1,\ \mu = 20. \tag{35}$$

Let the tracking signal be $r_t = 10 \sin t$ for all t.

Example 1. Random disturbances. Let the unknown parameters in the description (35) have the values

$$c^w = 5,\ \delta^w = 1,\ \delta^y = \delta^u = 0.1, \tag{36}$$

and let w_t, k_t^y, k_t^u be independent pseudorandom variables uniformly distributed on $[-1, 1]$. The simulation was performed with the following adaptive control parameters: the dead zone parameter $\varepsilon_1 = 10^{-6}$, $c^w_{\min} = -10$, $c^w_{\max} = 10$, and the grid step $\varepsilon_2 = 0.5$.

Figure 1 shows the graphs of the tracking error $y - r$ on the left and the current optimal control criterion estimates $J(c^w_{k_t}, \delta_{k_t,t})$ on the right.

Next, the final values $J(c^w_k, \delta_{k,1000})$ for all k, consistent with measurements on the interval [1, 1000], are presented in Fig. 2 on the left. The switching of the estimates $c^w_{k_t,t}$ of the unknown bias $c^w = 5$ are provided in Fig. 2 on the right. Despite the symmetry of the distributions of the random variables w_t, k_t^y, k_t^u about zero, the steady-state bias estimate $c^w_{1000} = 4.5$ differs from $c^w = 5$.

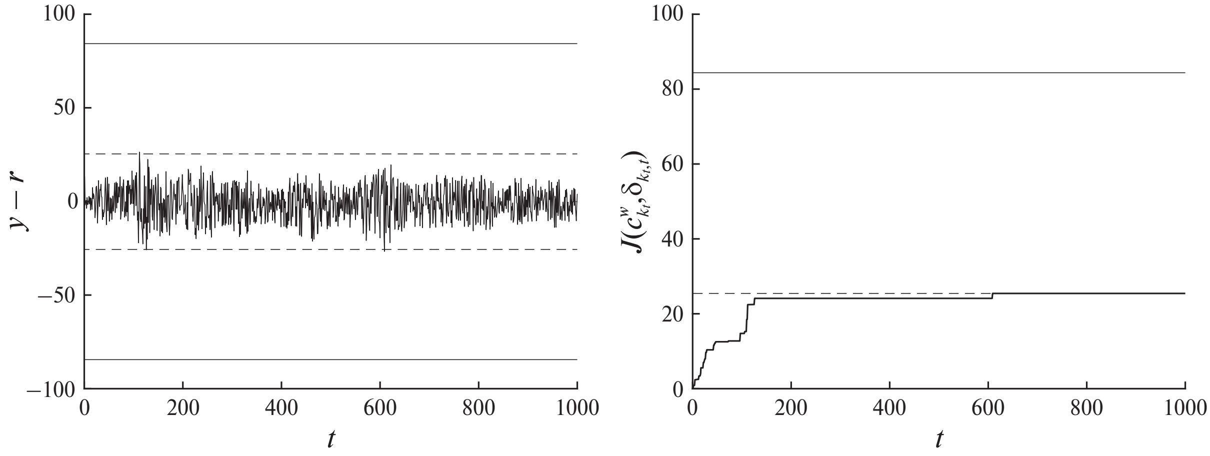

Fig. 1. The graphs of $y - r$ (left) and $J(c^w_{k_t}, \delta_{k_t,t})$ (right). Solid lines correspond to $\pm J(c^w, \delta)$ and dashed lines to $\pm J(c^w_{k_\infty}, \delta_{k_\infty})$.

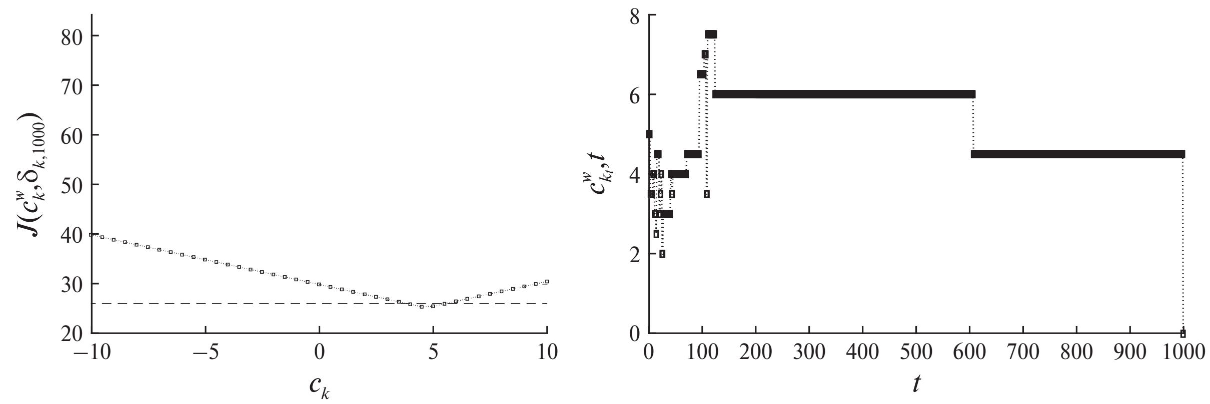

Fig. 2. The values $J(c^w_k, \delta_{k,1000})$ (left) and the switching of the estimates $c^w_{k_t,t}$ (right).

In all the numerical experiments with random perturbations and deterministic "oscillatory" disturbances, the steady-state upper bounds of the tracking error $J(c^w_{k_\infty}, \delta_{k_\infty})$, consistent with the measurements, are significantly (several times) smaller than the unknown optimal upper bound of $J(c^w, \delta)$. When quantifying the errors, the perturbations do not manifest themselves in any way since the current estimates δ_t of the unknown vector δ usually have the form $\delta_t = (c^w_t, 0, 0)$.

Remark 2. Proponents of stochastic disturbance models in system identification theory constantly criticize the set-membership approach for its seemingly inevitable conservatism due to the necessary a priori information about upper bounds on deterministic disturbances. (Here, only the conservatism of the set estimators of unknown parameters is implied.) As illustrated by Example 1, the use of set-membership estimation and the control criterion as the identification criterion makes the measurement-consistent performance guarantees non-conservative and, furthermore, improves performance guarantees compared to the optimal control criterion (5), since particular disturbance realizations are generally far from the disturbances maximizing the tracking error. This is analogous to the fact that in the stochastic case, average performance indices are better than the worst possible values on particular "bad" realizations. However, in problems with stochastic disturbances, disturbance model verification is usually not discussed. The optimality of tracking within the deterministic ℓ_1-theory is based on the verification of the disturbance model and the use of sufficiently complete information about unknown parameters obtained in the control process, and the price for optimality is a corresponding increase in the volume of necessary computations.

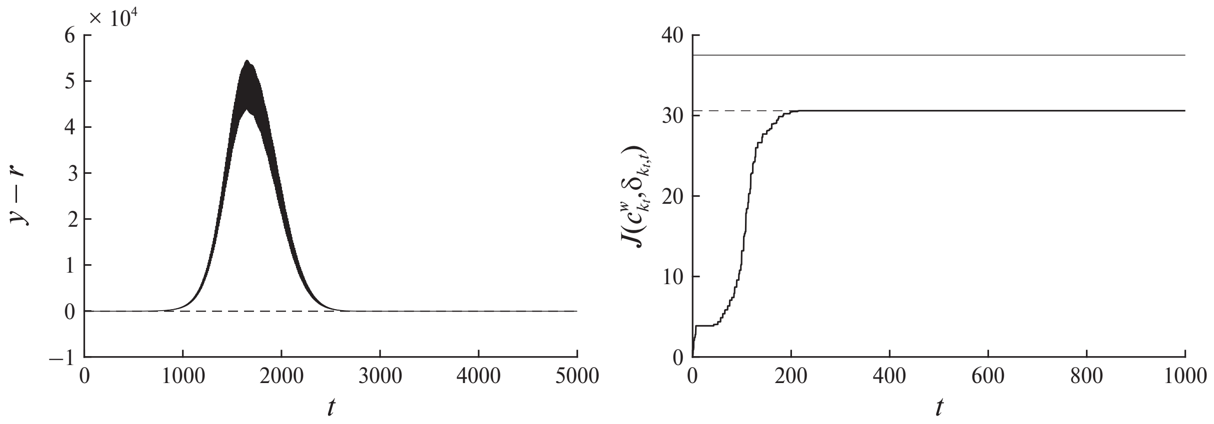

Fig. 3. The graphs of $y-r$ (left) and $J(c^w_{k_t}, \delta_{k_t,t})$ (right). Solid line corresponds to $J(c^w, \delta)$ and dashed line to $J(c^w_{k_\infty}, \delta_{k_\infty})$.

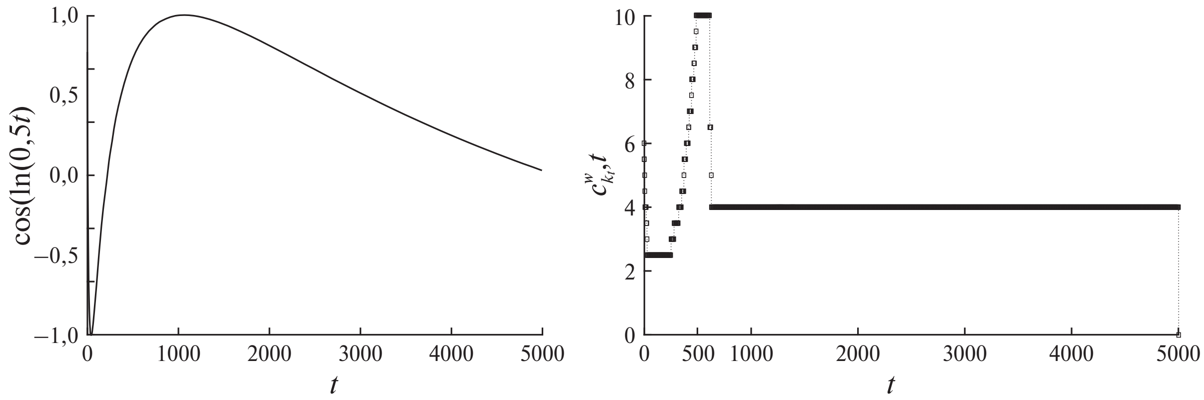

Fig. 4. The graphs of k^y_t and switching of the estimates of the unknown bias $c^w = 5$.

Example 2. "Bad" deterministic disturbances. This example is intended to demonstrate a "bad" total disturbance under which the presence of perturbations in the total disturbance v becomes evident.

Consider the plant (33) with the total disturbance (35) and the parameters (36) with the reduced value $\delta^u = 0.05$ and the deterministic sequences

$$w_t = \cos(50t),\ k^y_t = \sin(70t),\ k^u_t = \cos(\ln(0.5t)). \tag{37}$$

The left graph in Fig. 3 shows the tracking error of $y-r$. Under this disturbance, for all $t \geqslant 498$, the estimates are $\delta^u_{k_t,t} > 0$ and $\delta^y_{k_t,t} = 0$; the last estimates are $\delta_{k_{5000},5000} = (1.6993; 0; 0.0581)$ and $c^w_{5000} = 4$. Thus, starting from the time instant $t = 498$, the perturbation affecting the control manifests itself in the estimates $\delta_{k_t,t}$ of the disturbance norms.

As is known, stable linear time-invariant systems may have large deviations from zero due to nontrivial initial conditions or a bounded disturbance [16, 17]. In Example 2, a large deviation of the tracking error (with $\max_t |y_t - r_t| = 5.4573 \times 10^4$) in the nonlinear closed-loop system (1)–(6) can be caused both by the switching of the controllers corresponding to different estimates of the biases and by the "asymmetry" of the sequence $k^u_t = \cos(\ln(0.5t))$ about zero (see the left graph in Fig. 4). The current optimal estimates $c^w_{k_t,t}$ of the unknown bias c^w are shown in the right graph of Fig. 4, where the steady-state bias estimate is $c^w_{k_\infty} = 4 \neq c^w = 5$.

Remark 3. Despite the tracking error values in the transient mode having the order of magnitude 10^4 (unacceptable in applications), the asymptotic behavior of the tracking error under this disturbance is characterized by the numbers

$$\max_{t\in[4001,5000]} |y_t - r_t| = 4.4163, \qquad \max_{t\in[4901,5000]} |y_t - r_t| = 2.7566.$$

Thus, the factual steady-state tracking error is by an order of magnitude smaller than the final guaranteed tracking error estimate $J(c^w_{k_\infty}, \delta_{k_\infty}) = 30.4421$, consistent with the measurement data. In turn, this estimate is better than the optimal (but unknown!) value $J(c^w, \delta) = 37.2971$ guaranteed by Theorem 1, despite the low chosen "accuracy" of the bias estimates (the grid size $\varepsilon_2 = 0.5$). Finally, the optimal value $J(c^w, \delta)$ itself is less than the worst possible asymptotic tracking error since $J(c^w, \delta)$ ignores that the tested plant has the nominal parameters θ^* instead of θ. As a result, the adaptive compensation algorithm for the unknown bias c^w fulfills its purpose despite possible large deviations of the tracking error from zero and even "adapts" to particular realizations of the disturbance v, reducing excessive conservatism in the guaranteed performance estimates under non-"maximizing" disturbances.

Remark 4. According to the above graphs of the switching of the estimates $c^w_{k_t}$, the unknown bias c^w is non-identifiable in the description (4) even in the absence of perturbations since control always deals with particular realizations of the disturbance v_t for which the biases (for any reasonable definition) can be (more correctly, will be) different. That is, the term "bias" with respect to the constant c^w in the description (4) refers precisely to the concept of bias for the class of all disturbances satisfying (4). At the same time, the current estimates $c^w_{k_t}$ can (or rather should) be considered a correct definition (in the context of the control problem being solved) of the current estimates of the bias for a particular realization of the total disturbance v.

Remark 5. The volume and speed of computations in the above examples are characterized by the following indicators. The computation time on a laptop with 15.2 GB RAM and an Intel Core Ultra 5 125H processor is 2.99 s in Example 1 and 15.1169 s in Example 2. The number of inequalities in the polyhedral estimators $P_{k,t}$ is 12–15 in Example 1 and 64–81 in Example 2. The ratio of these limits approximately matches that of the interval lengths, equal to 5. The indicators of Example 2 on the time interval [1, 10 000] remain the same, meaning that the transient processes for the particular disturbance v under consideration have already been completed by the time instant $t = 5000$. Note that the computation time is determined mainly by the time to calculate the polyhedral estimators $P_{k,t}$ and the optimal vector estimators $\delta_{k,t}$ in $\mathbb{R}^3$ and is almost independent of the dimension of the nominal parameter vector θ.

Remark 6. The number of inequalities in the description of the polyhedral estimators $P_{k,t}$ and, as a consequence, the computation time of the optimal estimates in (21) can be reduced by eliminating possible redundant inequalities after adding the new inequalities (17); for details, see [18]. In the simulation results presented, this was not done in order to demonstrate the number of possible estimator updates even for a very small value of the dead zone parameter ε_1 (almost zero from the viewpoint of assessing the model quality).

6. CONCLUSIONS

This paper has considered a discrete minimum-phase plant with a known or specified nominal model (for testing), a biased and bounded external disturbance, and perturbations with unknown norms and an unknown bias. For this plant, the optimal tracking problem of a given bounded signal with a given accuracy has been addressed. The problem difficulty lies in the need to compensate for the bias based on reasonable optimal estimation of control performance under the non-identifiability of all unknown parameters. The solution of this problem involves errors quantification, set-membership estimation of unknown parameters, and the use of the control criterion as an ideal identification criterion. Within such an approach, it becomes possible to more deeply understand, demonstrate, and implement the maximum capabilities of feedback control. The importance of feedback research was emphasized by L. Guo, a leading expert in adaptive control and identification of systems, in the abstract of his paper [19]:

> "The main purpose of adaptive feedback is to deal with dynamical systems with internal and/or external uncertainties, by using the on-line observed information. Thus, a fundamental problem in adaptive control is to understand the maximum capability and limits of adaptive feedback."

In this context, we also provide a quotation from the abstract of his another paper [20]:

> "Finally, we will consider more fundamental problems on the maximum capability and limitations of the feedback mechanism in dealing with uncertain nonlinear systems, where the feedback mechanism is defined as the class of all possible feedback laws."

The solution presented in this paper not only ensures, with a given accuracy, the same tracking performance estimate as under the known parameters of the nominal plant and disturbances, but also gives significantly better guaranteed performance estimates depending on particular realizations of deterministic disturbances. Thus, it is implicitly considered that particular realizations of disturbances are usually far from those maximizing the control criterion: in order to maximize the tracking error estimate, the total disturbance v_t must not only take maximum values on a long time interval but also have definite signs on this interval.

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